

ON INSTANTON HOMOLOGY OF CORKS W_n

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ABSTRACT. We consider a family of corks, denoted W_n , constructed by Akbulut and Yasui. Each cork gives rise to an exotic structure on a smooth 4-manifold via a twist τ on its boundary $\Sigma_n = \partial W_n$. We compute the instanton Floer homology of Σ_n and show that the map induced on the instanton Floer homology by $\tau : \Sigma_n \rightarrow \Sigma_n$ is non-trivial.

1. INTRODUCTION

In [4], Akbulut and Yasui defined a cork C as a compact Stein 4-manifold with boundary together with an involution $\tau : \partial C \rightarrow \partial C$ which extends as a self-homeomorphism of C but not as a self-diffeomorphism. In addition, $C \subset X$ is a cork of a smooth 4-manifold X if cutting C out and regluing it via τ changes the diffeomorphism type of X .

We will consider the family of corks W_n , $n \geq 1$, obtained by surgery on the link in Figure 1 where a positive integer m in a box indicates m right-handed half-twists. The boundary Σ_n of W_n is the integral homology 3-sphere with surgery description as in Figure 2. The involution $\tau : \Sigma_n \rightarrow \Sigma_n$ interchanges the two components of the link in Figure 2. It is best seen when the underlying link L_n is drawn symmetrically, as in Figure 3. Note that the quotient manifold $\Sigma'_n = \Sigma_n/\tau$ is homeomorphic to S^3 so Σ_n can be viewed as a double branched cover of S^3 with branch set k_n as shown in Figure 4.

The goal of this paper is to study the instanton Floer homology $I_*(\Sigma_n)$ and the map $\tau_* : I_*(\Sigma_n) \rightarrow I_*(\Sigma_n)$ induced on it by τ .

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Theorem. (1) *For every integer $n \geq 1$, the instanton Floer homology group $I_j(\Sigma_n)$, $j = 0, \dots, 7$, is trivial if j is even, and is a free abelian group of rank $n(n+1)(n+2)/6$ if j is odd.*

(2) *The homomorphism $\tau_* : I_*(\Sigma_n) \rightarrow I_*(\Sigma_n)$ is non-trivial for all $n \geq 1$.*

The first example of an involution acting non-trivially on the instanton Floer homology of an irreducible homology 3-sphere was given in [1] and [16]; in fact, that example was exactly our $\tau : \Sigma_1 \rightarrow \Sigma_1$. The technique we use to show non-triviality of τ_* is the same as the technique that was used in [15] to reprove the result of [16]: compare the Lefschetz number of $\tau_* : I_*(\Sigma_n) \rightarrow I_*(\Sigma_n)$ with the Lefschetz number of the identity map. If the two are different, then the involution must be non-trivial. For any integral homology 3-sphere Σ , the Lefschetz number of the identity equals the Euler characteristic of $I_*(\Sigma)$, which by Taubes [18] is twice the Casson invariant $\lambda(\Sigma)$. Ruberman and Saveliev [14] showed that the Lefschetz number of τ_* equals twice the equivariant Casson invariant $\lambda^\tau(\Sigma)$, defined in [7]. Therefore the non-triviality of τ_* will follow as soon as we show that $\lambda(\Sigma_n) \neq \lambda^\tau(\Sigma_n)$. The calculation of $I_*(\Sigma_n) \rightarrow I_*(\Sigma_n)$ is done using surgery techniques.

It should be noted that Akbulut and Karakurt proved a Heegaard Floer analogue of this result. In [2] they showed that the involution $\tau : \Sigma_n \rightarrow \Sigma_n$ acts non-trivially on the Heegaard Floer homology group $HF^+(\Sigma_n)$.

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2. THE INSTANTON FLOER HOMOLOGY OF Σ_n

Let Σ be an oriented integral homology 3-sphere. The instanton homology groups $I_*(\Sigma)$ are eight abelian groups arising as the Floer homology of the Chern-Simons functional on the space of irreducible $SU(2)$ connections on Σ modulo gauge equivalence. The Euler characteristic of $I_*(\Sigma)$ is twice the Casson invariant $\lambda(\Sigma)$, see Taubes [18].

Let $\Sigma_n(p)$ be the integral homology sphere with surgery description obtained by replacing the 0-framing of the unknot on the left-hand side of

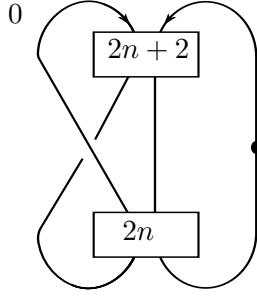


FIGURE 1. W_n

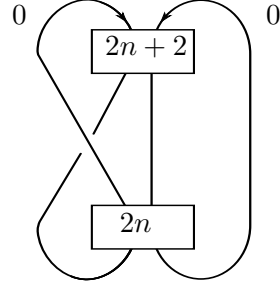


FIGURE 2. Σ_n

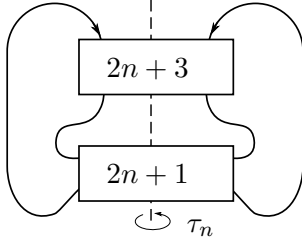


FIGURE 3. L_n

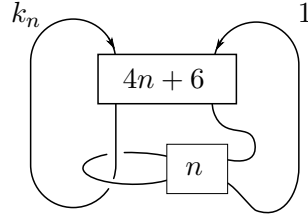


FIGURE 4. S^3

Figure 2 by a p -framing. In [16], Saveliev used the Floer exact triangle to show that the instanton homology groups of $I_*(\Sigma_1)(p)$ are independent of p . The same argument holds for $I_*(\Sigma_n)(p)$.

The homology 3-sphere $\Sigma_n(2n+2)$ was shown by Maruyama [13] to be homeomorphic to the Brieskorn homology sphere $\Sigma(2n+1, 2n+2, 2n+3)$. In Theorem 10 of [17], Saveliev proved that the Floer homology group $I_j(\Sigma(2n+1, 2n+2, 2n+3))$ is trivial when j is even, and is isomorphic to a free abelian group of rank $n(n+1)(n+2)/6$ when j is odd. This completes the calculation of $I_*(\Sigma_n)$. In particular, the Casson invariant of Σ_n is given by

$$\lambda(\Sigma_n) = -\frac{1}{3}n(n+1)(n+2).$$

3. THE CASSON INVARIANT

Although the instanton Floer homology groups of Σ_n are known, and therefore so is its Casson invariant, we can compute $\lambda(\Sigma_n)$ using topological methods.

Let L be a framed 2-component link in S^3 , and assume that the 3-manifold Σ resulting from surgery on L is a homology 3-sphere. Boyer and Lines [5] compute $\lambda(\Sigma)$ as a sum of derivatives of the multivariable Alexander polynomial Δ_L of the underlying oriented link L (and those of its sublinks), and Dedekind sums that only depend on the framings of the link components.

In our case, both components of L_n are framed by zero, and the Boyer-Lines [5] formula for $\lambda(\Sigma_n)$ is simply

$$\lambda(\Sigma_n) = -\frac{1}{\det(B)} \frac{\partial^2 \Delta_{L_n}}{\partial x \partial y}(1, 1), \quad (1)$$

where B is the framing matrix for Σ_n . Thus to compute $\lambda(\Sigma_n)$ we will need only to compute Δ_{L_n} .

3.1. The Alexander polynomial and Conway potential function.

Rather than computing Δ_L directly we will consider the related Conway potential function ∇_L . Given an oriented link L in S^3 , normalize $\Delta_L(x, y)$ using the Conway potential function $\nabla_L(x, y)$ of Hartley [11] by requiring that

$$\Delta_L(x^2, y^2) = \nabla_L(x, y). \quad (2)$$

Note that (2) implies that

$$\frac{\partial^2 \Delta_L}{\partial x \partial y}(1, 1) = \frac{1}{4} \frac{\partial^2 \nabla_L}{\partial x \partial y}(1, 1), \quad (3)$$

hence we only need to compute the Conway potential function of L and its partial derivatives.

The Conway potential function ∇_L enjoys the replacement relation

$$\nabla_\ell + \nabla_r = (xy + x^{-1}y^{-1})\nabla_s, \quad (4)$$

where ∇_ℓ , ∇_r , and ∇_s are Conway functions of links that differ only in a neighborhood of a single crossing as shown in Figure 5. Note that the arcs may belong to the same component of L or to different components. If the

three links differ as in Figure 5 but with one of the arcs oppositely oriented, then we have the relation

$$\nabla_\ell + \nabla_r = (xy^{-1} + x^{-1}y)\nabla_s. \quad (5)$$

Also, we will note that the Conway potential function vanishes for split links and is equal to 1 for the right-handed Hopf link.

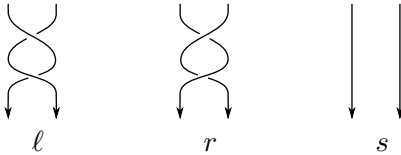


FIGURE 5.

In order to calculate the Conway potential function of L_n we will use the replacement relations (4) and (5) to produce a linear recurrence which we will then solve.

3.2. The recurrence relation. Let $f_n = \nabla_{L_n}$ and $g_n = \nabla_{H_n}$, where H_n is the link in Figure 6. A straightforward calculation shows that

$$g_n = \left(\frac{1}{r_2 - r_1} \right) r_1^n + \left(\frac{1}{r_1 - r_2} \right) r_2^n$$

with $v = xy^{-1} + x^{-1}y$, $r_1 = (v + \sqrt{v^2 - 4})/2$, and $r_2 = (v - \sqrt{v^2 - 4})/2$. Using Hartley's replacement relations (4), we change crossings of L_n two at a time, until we have undone the upper tangle in Figure 2. We obtain the recurrence $f_{n+2} = -f_n + uf_{n+1}$ with initial conditions $f_0 = -g_{n+1} + ug_n$ and $f_1 = -ug_{n+1} + (u^2 - 1)g_n$, where $u = xy + x^{-1}y^{-1}$.

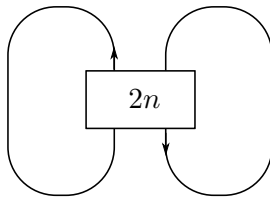


FIGURE 6. H_n

Solving the recurrence relation, we obtain a formula for f_n in terms of x, y , and n ,

$$f_n = \left(f_0 + \frac{f_0 s_1 - f_1}{s_2 - s_1} \right) s_1^n + \left(\frac{f_1 - f_0 s_1}{s_2 - s_1} \right) s_2^n$$

where $s_1 = (u + \sqrt{u^2 - 4})/2$ and $s_2 = (u - \sqrt{u^2 - 4})/2$. We then find an explicit formula for $\lambda(\Sigma_n)$ by having *Maple* [12] differentiate f_n twice and putting together (1) and (3). The answer is

$$\lambda(\Sigma_n) = -\frac{1}{3}n(n+1)(n+2). \quad (6)$$

4. THE EQUIVARIANT CASSON INVARIANT

Let $\tau : \Sigma \rightarrow \Sigma$ be an orientation preserving involution on a homology 3-sphere, and suppose that the fixed point set of τ is non-empty. Then the quotient manifold $\Sigma' = \Sigma/\tau$ is a homology 3-sphere, and the projection map $\Sigma \rightarrow \Sigma'$ is a double branched cover with branch set a knot $k \subset \Sigma'$. In [7], Collin and Saveliev computed the equivariant Casson invariant $\lambda^\tau(\Sigma)$ in terms of $\lambda(\Sigma')$ and the knot signature $\sigma(k)$. When $\Sigma' = S^3$, we have simply

$$\lambda^\tau(\Sigma) = \frac{1}{8}\sigma(k).$$

Since we know that $\lambda(\Sigma_n)$ is decreasing as $n \rightarrow \infty$, see (6), our strategy for showing that $\lambda^\tau(\Sigma_n) \neq \lambda(\Sigma_n)$ will be to show that $\sigma(k_n)/8$ is bounded from below by a function strictly greater than $\lambda(\Sigma_n)$, where k_n is the knot shown in Figure 4.

4.1. Bounding knot signatures. The knot signature of a knot $k \subset S^3$ may be bounded from below using the formula

$$\sigma(k_r) \leq \sigma(k_\ell) \leq \sigma(k_r) + 2, \quad (7)$$

see Conway [8] or Giller [10]. Here k_r and k_ℓ are knots that only differ in a neighborhood of a crossing as shown in Figure 7. Note that our sign convention is opposite of Giller's. By (7), the signature of a knot k is bounded from below by negative twice the number of right-handed crossings that must be changed in order to undo the knot k . Note that we may need to change some left handed crossings while undoing k but this will not contribute to our estimate.

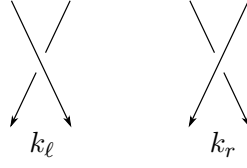


FIGURE 7.

We will now apply this observation to the knot k_n shown in Figure 4. In order to see k_n more clearly, we will isotope the link diagram in Figure 4 so that the 1-framed curve is interchanged with the branch set k_n and then blow down the 1-framed curve. The blow down has the effect of a full left-handed twist on the $2n + 4$ strands passing through the 1-framed curve, see Figure 8.

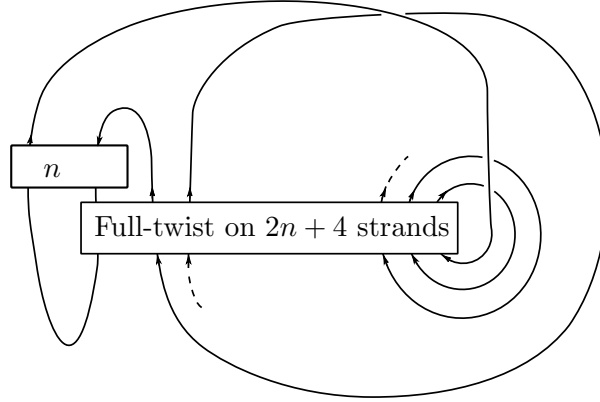


FIGURE 8. k_n

Note that when n is odd, Figure 8 is a diagram of k_n where no right-handed crossings must be changed in order to undo the knot, hence $\sigma(k_n)$ must be non-negative. When n is even, the left-most strand of the strands being twisted in Figure 8 is oriented oppositely of the other strands being twisted. In this case there are $2n + 3$ right-handed crossings that must be changed. We change them to arrive at a knot with only left-handed crossings that further need to be changed. Thus the signature of k_n is bounded from below by $-4n - 6$, and we have shown that

$$\lambda(\Sigma_n) = -\frac{n(n+1)(n+2)}{3} < -\frac{4n+6}{8} \leq \lambda^\tau(\Sigma_n).$$

Lastly, we remark that for small n the equivariant Casson invariant can be computed explicitly using the following technique. If we undo the upper left tangle in Figure 8 by changing crossings, then we will arrive at a knot that is isotopic to a torus knot. If n is odd, then the corresponding torus knot is the $T(2n+4, 2n+3)$ torus knot. If n is even, then the corresponding torus knot is $T(2n+2, 2n+1)$. If $n \leq 4$, then we need to change at most 3 crossings, and consequently the signature of k_n differs from that of the corresponding torus knot by at most ± 6 .

For example, if $n = 1$ or $n = 2$, then the corresponding torus knot is $T(6, 5)$ and the signature $\sigma(T(6, 5))$ is 16. Since the signature of k_n is divisible by 8, we have that $\sigma(k_1) = \sigma(k_2) = 16$. Similarly, if $n = 3$ or $n = 4$, then the corresponding torus knot is $T(10, 9)$ and

$$\sigma(k_3) = \sigma(k_4) = \sigma(T(10, 9)) = 48.$$

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